Hidden Markov Models for Time Series
An Introduction Using R
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Für Hanne und Werner,
mit herzlichem Dank für Eure Unterstützung
bei der Suche nach den versteckten Ketten.
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Preface

In the eleven years since the publication of our book *Hidden Markov and Other Models for Discrete-valued Time Series* it has become apparent that most of the ‘other models’, though undoubtedly of theoretical interest, have led to few published applications. This is in marked contrast to hidden Markov models, which are of course applicable to more than just discrete-valued time series. These observations have led us to write a book with different objectives.

Firstly, our emphasis is no longer principally on discrete-valued series. We have therefore removed Part One of the original text, which covered the ‘other models’ for such series. Our focus here is exclusively on hidden Markov models, but applied to a wide range of types of time series: continuous-valued, circular, multivariate, for instance, in addition to the types of data we previously considered, namely binary data, bounded and unbounded counts and categorical observations.

Secondly, we have attempted to make the models more accessible by illustrating how the computing environment R can be used to carry out the computations, e.g., for parameter estimation, model selection, model checking, decoding and forecasting. In our previous book we used proprietary software to perform numerical optimization, subject to linear constraints on the variables, for parameter estimation. We now show how one can use standard R functions instead. The R code that we used to carry out the computations for some of the applications is given, and can be applied directly in similar applications. We do not, however, supply a ready-to-use package; packages that cover ‘standard’ cases already exist. Rather, it is our intention to show the reader how to go about constructing and fitting application-specific variations of the standard models, variations that may not be covered in the currently available software. The programming exercises are intended to encourage readers to develop expertise in this respect.

The book is intended to illustrate the wonderful plasticity of hidden Markov models as general-purpose models for time series. We hope that readers will find it easy to devise for themselves ‘customized’ models that will be useful in summarizing and interpreting their data. To this end we offer a range of applications and types of data — Part Two is
entirely devoted to applications. Some of the applications appeared in
the original text, but these have been extended or refined.

Our intended readership is applied statisticians, students of statistics,
and researchers in fields in which time series arise that are not amenable
to analysis by the standard time series models such as Gaussian ARMA
models. Such fields include animal behaviour, epidemiology, finance, hy-
drology and sociology. We have tried to write for readers who wish to
acquire a general understanding of the models and their uses, and who
wish to apply them. Researchers primarily interested in developing the
theory of hidden Markov models are likely to be disappointed by the
lack of generality of our treatment, and by the dearth of material on
specific issues such as identifiability, hypothesis testing, properties of es-
timators and reversible jump Markov chain Monte Carlo methods. Such
readers would find it more profitable to refer to alternative sources, such
as Cappé, Moulines and Rydén (2005) or Ephraim and Merhav (2002).
Our strategy has been to present most of the ideas by using a single run-
ning example and a simple model, the Poisson–hidden Markov model.
In Chapter 8, and in Part Two of the book, we illustrate how this basic
model can be progressively and variously extended and generalized.

We assume only a modest level of knowledge of probability and statis-
tics: the reader is assumed to be familiar with the basic probability distri-
butions such as the Poisson, normal and binomial, and with the concepts
of dependence, correlation and likelihood. While we would not go as far
as Lindsey (2004, p. ix) and state that ‘Familiarity with classical intro-
ductive statistics courses based on point estimation, hypothesis testing,
confidence intervals […] will be a definite handicap’, we hope that exten-
sive knowledge of such matters will not prove necessary. No prior knowl-
edge of Markov chains is assumed, although our coverage is brief enough
that readers may wish to supplement our treatment by reading the rel-
vant parts of a book such as Grimmett and Stirzaker (2001). We have
also included exercises of a theoretical nature in many of the chapters,
both to fill in the details and to illustrate some of the concepts intro-
duced in the text. All the datasets analysed in this book can be accessed
at the following address: http://134.76.173.220/hmm-with-r/data.

This book contains some material which has not previously been pub-
lished, either by ourselves or (to the best of our knowledge) by others.
If we have anywhere failed to make appropriate acknowledgement of
the work of others, or misquoted their work in any way, we would be
grateful if the reader would draw it to our attention. The applications de-
scribed in Chapters 14, 15 and 16 contain material which first appeared in
(respectively) the South African Statistical Journal, the International
Journal of Epidemiology and Biometrics. We are grateful to the editors
of these journals for allowing us to reuse such material.

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We wish to thank the following researchers for giving us access to their data, and in some cases spending much time discussing it with us: David Bowie, Graham Fick, Linda Haines, Len Lerer, Frikkie Potgieter, David Raubenheimer and Max Suster.

We are especially indebted to Andreas Schlegel and Jan Bulla for their important inputs, particularly in the early stages of the project; to Christian Gläser, Oleg Nenadić and Daniel Adler, for contributing their computing expertise; and to Antony Unwin and Ellis Pender for their constructive comments on and criticisms of different aspects of our work. The second author wishes to thank the Institute for Statistics and Econometrics of Georg–August–Universität, Göttingen, for welcoming him on many visits and placing facilities at his disposal. Finally, we are most grateful to our colleague and friend of many years, Linda Haines, whose criticism has been invaluable in improving this book.

Göttingen
November 2008
## Notation and abbreviations

Since the underlying mathematical ideas are the important quantities, no notation should be adhered to slavishly. It is all a question of who is master.

Bellman (1960, p. 82)

[... ] many writers have acted as though they believe that the success of the Box–Jenkins models is largely due to the use of the acronyms.

Granger (1982)

### Notation

Although notation is defined as it is introduced, it may also be helpful to list here the most common meanings of symbols, and the pages on which they are introduced. Matrices and vectors are denoted by bold type. Transposition of matrices and vectors is indicated by the prime symbol: 

All vectors are row vectors unless indicated otherwise.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(i) )</td>
<td>( i )th column of any matrix ( A )</td>
<td>86</td>
</tr>
<tr>
<td>( A_n(\kappa) )</td>
<td>( I_\kappa(\kappa)/I_0(\kappa) )</td>
<td>160</td>
</tr>
<tr>
<td>( B )</td>
<td>( \Gamma P(x_t) )</td>
<td>37</td>
</tr>
<tr>
<td>( C_t )</td>
<td>state occupied by Markov chain at time ( t )</td>
<td>16</td>
</tr>
<tr>
<td>( C^{(t)} )</td>
<td>( (C_1, C_2, \ldots, C_t) )</td>
<td>16</td>
</tr>
<tr>
<td>( {g_t} )</td>
<td>parameter process of a stochastic volatility model</td>
<td>190</td>
</tr>
<tr>
<td>( I_n )</td>
<td>modified Bessel function of the first kind of order ( n )</td>
<td>156</td>
</tr>
<tr>
<td>( l )</td>
<td>log-likelihood</td>
<td>21</td>
</tr>
<tr>
<td>( L ) or ( LT )</td>
<td>likelihood</td>
<td>21, 35</td>
</tr>
<tr>
<td>( \log )</td>
<td>logarithm to the base ( e )</td>
<td></td>
</tr>
<tr>
<td>( n )</td>
<td>number of states in a Markov chain, or number of components in a mixture</td>
<td>7</td>
</tr>
<tr>
<td>( N )</td>
<td>the set of all positive integers</td>
<td>17</td>
</tr>
<tr>
<td>( N_t )</td>
<td>nutrient level</td>
<td>220</td>
</tr>
<tr>
<td>( N(\bullet; \mu, \sigma^2) )</td>
<td>distribution function of general normal distribution</td>
<td>191</td>
</tr>
<tr>
<td>( n(\bullet; \mu, \sigma^2) )</td>
<td>density of general normal distribution</td>
<td>191</td>
</tr>
<tr>
<td>( p_i )</td>
<td>probability mass or density function in state ( i )</td>
<td>31</td>
</tr>
<tr>
<td>( P(x) )</td>
<td>diagonal matrix with ( i )th diagonal element ( p_i(x) )</td>
<td>32</td>
</tr>
<tr>
<td>( R )</td>
<td>the set of all real numbers</td>
<td></td>
</tr>
</tbody>
</table>
NOTATION AND ABBREVIATIONS

$T$  length of a time series 35
$U$  square matrix with all elements equal to 1 19
$u(t)$  vector $(\Pr(C_t = 1), \ldots, \Pr(C_t = m))$ 17
$u_i(t)$  $\Pr(C_t = i)$, i.e. $i$th element of $u(t)$ 32
$w_t$  $\alpha_t \Gamma' = \sum_i \alpha_t(i)$ 46
$X_t$  observation at time $t$, or just $t$th observation 30
$X(t)$  $(X_1, X_2, \ldots, X_t)$ 30
$X(-t)$  $(X_1, \ldots, X_{t-1}, X_{t+1}, \ldots, X_T)$ 76
$X^b$  $(X_a, X_{a+1}, \ldots, X_b)$ 61
$\alpha_t$  (row) vector of forward probabilities 38
$\alpha_t(i)$  forward probability, i.e. $\Pr(X(t) = x(t), C_t = i)$ 59
$\beta_t$  (row) vector of backward probabilities 60
$\beta_t(i)$  backward probability, i.e. $\Pr(X^T_{t+1} = x^T_{t+1} | C_t = i)$ 60
$\Gamma$  transition probability matrix of Markov chain 17
$\gamma_{ij}$  $(i, j)$ element of $\Gamma$; probability of transition from state $i$ to state $j$ in a Markov chain 17
$\delta$  stationary or initial distribution of Markov chain, or vector of mixing probabilities 18
$\phi_t$  vector of forward probabilities, normalized to have sum equal to 1, i.e. $\alpha_t/w_t$ 46
$\Phi$  distribution function of standard normal distribution 19
$1$  (row) vector of ones 19

Abbreviations

ACF  autocorrelation function
AIC  Akaike’s information criterion
BIC  Bayesian information criterion
CDLL  complete-data log-likelihood
c.o.d.  change of direction
c.v.  coefficient of variation
HM  hidden Markov
HMM  hidden Markov model
MC  Markov chain
MCMC  Markov chain Monte Carlo
ML  maximum likelihood
MLE  maximum likelihood estimator or estimate
PACF  partial autocorrelation function
qq-plot  quantile-quantile plot
SV  stochastic volatility
t.p.m.  transition probability matrix

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